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Classification of networks of automata by dynamical mean-field theory

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Abstract. Dynamical mean-field theory is used to classify the $2^{2^4} = 65\,536$ different networks of binary automata on a square lattice with nearest-neighbour interactions. Application of mean-field theory gives 700 different mean-field classes, which fall in seven classes of different asymptotic dynamics characterised by fixed points and 2-cycles.

1. Introduction

Networks of binary automata are the simplest extended, fully non-linear dynamical systems imaginable. Despite the very simple properties of the individual automaton, networks of these primitive elements possess a highly complex dynamics, demonstrating both regular and chaotic behaviour, separated by well defined phase transitions in the parameter space of automaton rules.

Regular as well as random networks of *random* automata have been extensively studied and their dynamical properties are rather well understood [1-12]. For random networks analytical results have been obtained for a number of properties, some of which were shown to be universal properties of disparate disordered systems ranging from spin glasses to randomly broken objects [6-8].

Regular networks of homogenous (defined below) automata is a larger field, and less well studied. One-dimensional networks were classified according to phenotypes by Wolfram some years ago ([13], reprinted in [14]). (A *phenotypic* classification is based on observed behaviour, while a *genotypic* classification is based on intrinsic properties of the entities being classified.) Many special cases of two-dimensional networks have been studied for their ability to model physical, chemical and biological systems ([14] and references therein). An explanatory phenomenological study of two-dimensional homogenous networks was reported and many open questions formulated in [15], reprinted in [14]. Much work has been done to develop a so-called local structure theory for the purpose of genotypic classification of regular networks [16-18]. This is an approximation scheme which may be applied to various orders, higher orders giving better approximations, in general. In [17] the local structure theory was applied to the equivalent of Conway's *Game of Life* on a hexagonal lattice. With this exception it has only been applied to one-dimensional networks and its

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systematic application to any order is not obvious in dimensions higher than one. To order one, local structure theory is just mean-field theory, and may be applied in any dimension. The higher the dimension, the better the approximation it gives.

Stauffer has recently undertaken the classification of two-dimensional, homogenous networks of automata according to their limit behaviour as time goes to infinity. This project is the two-dimensional equivalent of the classification of one-dimensional networks by Wolfram, and a phenotypic classification. The higher dimension makes Stauffer's project vastly more demanding, as is amply illustrated by preliminary results for automata on square lattices receiving inputs only from their four nearest neighbours [19]. The number of binary automata receiving four inputs is $2^{2^4} = 65\,536$. The eightfold D(4) symmetry of a square lattice with 'spins' that can be either 'up' or 'down' reduces this number by a factor slightly less than eight, leaving more than 8 342 different networks of automata to classify. The limit behaviour of the magnetisation of all these networks must be described as a function of the initial magnetisation. Because of the highly nonlinear dynamics of these systems, the only exact way to do this is in general to *compute* the time-development for enough initial magnetisations, until late enough times, on lattices large enough not to show noticeable finite-size effects. This must be done for more than 8 000 different networks. Clearly a formidable task no matter how one defines 'late enough' and 'large enough'! Consequently, any indication of what results to expect is of value.

In the present article we use dynamical mean-field theory to describe the time evolution of the magnetisation of each of the 65 536 different binary automata receiving four inputs, and classify them according to their limit behaviour. The mean-field description treats several automata as identical, including those that *are* identical because of the D(4) symmetry of the square lattice. Thus the mean-field description gives, by its mere application, a first classification of the 65 536 automata into 700 different classes (section 2). For each of these classes the time evolution of the magnetisation $m(t)$ is given by a fourth-degree polynomial mapping $m(t)$ into $m(t+1)$. These 700 polynomials are classified according to the properties of the map $m(t=0) \to m(t=\infty)$, i.e. according to the nature and location of fixed points and cycles of the map $m(t) \to m(t+1)$. That gives as a final result seven different classes with some subclassification (section 3). Since the time evolution of the networks studied here is given by two non-interacting subsystems, some subtleties must be sorted out before the mean-field predictions can be compared with simulation results (section 4). Section 5 contains our conclusions. The distribution of the original 65 536 networks over the initial 700 mean-field classes, the further classification of the latter, and fixed-point and cycle values for the mean-field classes have been tabulated, and are available from the third author (HF) in machine-readable form.

2. Dynamical mean-field theory

We consider a square lattice with a binary (or Boolean, or Ising spin) variable σ_i at each lattice site i. These dynamical variables develop in discrete time t according to a rule

$$\sigma_i(t+1) = f(\sigma_{i+\hat{1}}, \sigma_{i-\hat{1}}, \sigma_{i+\hat{2}}, \sigma_{i-\hat{2}}) \tag{1}$$

where $i \pm \hat{1}$ and $i \pm \hat{2}$ denote nearest-neighbour sites to site i along the first and second lattice axis, respectively. Since the binary function f is the same on all lattice sites the network is called homogenous. The magnetisation at time t is defined in the usual way as

$$m(t)=\frac{1}{N}\sum_{i=1}^{N}\sigma_i(t) \tag{2}$$

where N is the number of lattice sites, preferably taken to infinity.

This magnetisation is about the simplest conceivable measure of the activity in the network. A constant value of $m=1$ or -1 explains itself. Other values, constant or non-constant, contain too little information to fully determine the state of the network. But, focusing on the magnetisation alone, we may ask the following questions.

(i) Does the limit behaviour of $m(t)$ for $t\to\infty$ depend only on $m(0)$, or will two initial configurations $(\sigma_i(0))_{i=1\dots N}$ with the same magnetisation typically lead to different limit behaviours of their magnetisations? *Examples* of the latter are easily constructed on finite lattices. Our concern is with the question in the 'thermodynamical' limit of an infinite lattice.

(ii) In case of the first possibility, can the functional relation between $m(0)$ and the limit behaviour of $m(t)$ be determined for a given rule f?

(iii) Do several rules f give rise to similar limit behaviour of $m(t)$ as a function of $m(0)$, thus leading to a classification of the two-dimensional networks of automata considered here?

As anticipated in the introduction, within the mean-field description applied here the answers to these questions are as follows.

(i) $m(t+1)$ is a function of $m(t)$ only.

(ii) This function can be found for given $f(\sigma)$, and with it the limit behaviour sought for.

(iii) The networks considered can be classified according to similarities in their limit behaviour as derived from the map $m(t)\to m(t+1)$.

Mean-field theory is conceived as a theory for statistical averages, and hence is well suited for the problem just posed. It is an approximate method, however. Since it neglects fluctuations, it also neglects correlations between them. Consequently our results should be considered qualitative or semi-quantitative pointers to what to expect and look for in simulations. At the same time, our results may be considered a leading-order approximation in an expansion scheme that gradually takes correlations into account. This is the usual $1/(2d)$ expansion giving corrections to mean-field theory, $2d$ being the coordination number of a hyper-cubical lattice in d dimensions. The mean-field theory is derived in the following way. Using (1) and (2)

$$m(t+1)=\frac{1}{N}\sum_i f(\sigma_{i+\hat{1}}(t),\sigma_{i-\hat{1}}(t),\sigma_{i+\hat{2}}(t),\sigma_{i-\hat{2}}(t)). \tag{3}$$

We choose $\sigma_i(t)\in\{-1,1\}\forall i,t$, and call it a 'spin'. Then $\frac{1}{2}(1+\sigma m(t))$ is the fraction of spins on the lattice that have the value σ at time t. On the right-hand side of (3) we average over the lattice. If, as an approximation, we neglect correlations between the four next-neighbour spins $\sigma_{i+\hat{1}}(t)$, $\sigma_{i-\hat{1}}(t)$, $\sigma_{i+\hat{2}}(t)$ and $\sigma_{i-\hat{2}}(t)$, they independently take the values $\sigma=\pm1$ on the fraction $\frac{1}{2}(1+\sigma m(t))$ of all lattice sites at time t. This is the mean-field approximation. Used in (3) it gives

$$m'=T(m)=\tfrac{1}{16}\sum_{\sigma_1,\sigma_2,\sigma_3,\sigma_4=\pm1} f(\sigma_1,\sigma_2,\sigma_3,\sigma_4)(1+m\sigma_1)(1+m\sigma_2)(1+m\sigma_3)(1+m\sigma_4) \tag{4}$$

with $m = m(t)$ and $m' = m(t+1)$. Equation (4) is conveniently rewritten as

$$m' = T(m) = c_1\left(\frac{1-m}{2}\right)^4 + c_2\left(\frac{1-m}{2}\right)^3\left(\frac{1+m}{2}\right) + c_3\left(\frac{1-m}{2}\right)^2\left(\frac{1+m}{2}\right)^2 + c_4\left(\frac{1-m}{2}\right)\left(\frac{1+m}{2}\right)^3 + c_5\left(\frac{1+m}{2}\right)^4 \tag{5}$$

where

$$\begin{aligned}
c_1 &= f(-1,-1,-1,-1) && \in\{-1,1\}\\
c_2 &= \sum_{\sigma_1+\sigma_2+\sigma_3+\sigma_4=-2} f(\sigma_1,\sigma_2,\sigma_3,\sigma_4) && \in\{-4,-2,0,2,4\}\\
c_3 &= \sum_{\sigma_1+\sigma_2+\sigma_3+\sigma_4=0} f(\sigma_1,\sigma_2,\sigma_3,\sigma_4) && \in\{-6,-4,-2,0,2,4,6\}\\
c_4 &= \sum_{\sigma_1+\sigma_2+\sigma_3+\sigma_4=2} f(\sigma_1,\sigma_2,\sigma_3,\sigma_4) && \in\{-4,-2,0,2,4\}\\
c_5 &= f(1,1,1,1) && \in\{-1,1\}
\end{aligned} \tag{6}$$

are independent parameters with the ranges of values shown.

Equation (6) shows that there are $2\times5\times7\times5\times2=700$ different maps $T: m\to m'$ described in (5), and gives the map corresponding to any function f. The $2^{2^4}=65\,536$ different functions f are thus divided into 700 classes of functions which are indistinguishable in the mean-field description. In a given class, characterised by a set of allowed values for $(c_1, c_2, c_3, c_4, c_5)$, there are

$$N(c_1,c_2,c_3,c_4,c_5) = \frac{4!}{(2-|c_2|/2)!(2+|c_2|/2)!}\,\frac{6!}{(3-|c_3|/2)!(3+|c_3|/2)!} \times\frac{4!}{(2-|c_4|/2)!(2+|c_4|/2)!} \tag{7}$$

of the original 65 536 functions.

In the next section we classify the 700 different maps (5) according to the asymptotic behaviour of repeated application of a map to the interval $[-1, 1]$.

3. Solution and classification

Since in our mean-field description the time evolution of the network is obtained by iterating the map T in (5), the attractors—fixed points, cycles or strange—of this map are of prime interest. They represent the possible asymptotic patterns of behaviour of the system as time goes to infinity. Repulsive fixed points and cycles are of equal interest, because they separate the basins of attraction of the attractors, and in this sense are critical points on the axis of initial magnetisations.

Since the fixed-point equation

$$T(m^*) = m^* \tag{8}$$

determines m^* as a root in the fourth-degree polynomial $T(m)-m$, it can be determined analytically. The analytical expressions for the roots of a fourth-degree polynomial are, however, not very handy; and we also need the value of the derivative $\mathrm{d}T/\mathrm{d}m(m^*)$

at the fixed points, to determine whether they are attractive, marginal or repulsive. Furthermore, the equation for p-cyclic points

$$T^p(m^*) = m^* \tag{9}$$

determines m^* as roots in a $4p$-degree polynomial, hence they cannot be found analytically for $p > 1$.

A great many of the 700 maps T in question obviously have -1, 0 and/or 1 as fixed or cyclic points. In these cases the remaining points of interest are easier to find and characterise analytically. We found this simplification of little help in classifying all 700 maps. Even though these special cases are easier to handle, having to handle special cases is in itself a complication. Thus, considerations of intellectual economy favours application of brute force.

We let a computer program go through all 700 functions. For each function T it considered a large number of initial values m, equally spaced on the interval $[-1, 1]$, applied T a large number of times to each initial value, and checked whether it had become a fixed or cyclic point for T. In this way all attractors and repulsors were found for every function T, which was classified accordingly. In such a numerical approach it may be a problem to distinguish a strange attractor from a finite, but long, cycle. Since we found no cycles with periods larger than 2 among the 700 functions under consideration, we did not have this problem.

We find the classes shown in table 1 for the 700 maps defined in (4). The table also shows the number of functions f in (1) that fall in a given class by the mean-field approximation (4).

From the expression (5) for T we see a relationship

$$T(m; c_1, c_2, c_3, c_4, c_5) = -T(-m; -c_5, -c_4, -c_3, -c_2, -c_1) \tag{10}$$

holds between pairs of maps that are not odd with respect to m. Maps that *are* odd with respect to m are characterised by $(c_1, c_2, c_3, c_4, c_5) = (-c_5, -c_4, -c_3, -c_2, -c_1)$, and are only related to themselves by (10). From (10) it follows that if m^* is a fixed point of $T(\ ; c_1, c_2, c_3, c_4, c_5)$, then $-m^*$ is a fixed point of $-T(\ ; -c_5, -c_4, -c_3, -c_2, -c_1)$. Similarly, a 2-cycle (m_1, m_2) of the first map T causes the second map to have $(-m_2, -m_1)$ as 2-cycle. Fixed points or 2-cycles related this way are of the same nature, attractive or repulsive. Consequently, maps related by (10) belong to the same class A, B, C, D, E, F or G, but not necessarily to the same subclass. From the way sub-classes are characterised it follows that (10) relates all functions in a subclass either to functions in the same subclass (cases: A7, A8, B9, C3, C4, D, E, F, G) or to all functions in another subclass, i.e. (10) relates pairs of subclasses: A1 with A2, A3 with A4, A5 with A6, B5 with B6, B3 with B4, B7 with B8, B1 with B2, and C1 with C2.

These relationships and the special, but multitudinous, cases of functions with attractors and repulsors that are readily determined analytically, were used to check the results of the programs automatic classification. The special cases are as follows.

$c_1 = -1$: -1 is a fixed point. This result is exact, not just for the mean-field description, but also for the original dynamics (1).

$c_5 = 1$: 1 is a fixed point. This result is also exact for the original dynamics (1).

$c_1 + c_2 + c_3 + c_4 + c_5 = 0$: 0 is a fixed point.

When two of these three cases occur simultaneously, or -1, 0 or 1 is a marginal fixed point—i.e. a doublet root of $T(m) - m$—the remaining fixed points of T are

Table 1. The seven mean-field classes divided into subclasses. Column 1: name of (sub)class. Column 2: number of maps T in the (sub)class. Column 3: number of functions f falling in the (sub)class. Column 4: attractors by which the (sub)class is classified. Column 5: repulsors delimiting basins of attraction. Notation: m^* = fixed point; (m_1, m_2) = 2-cycle.

Class	Number of maps T	Number of functions f	Attractors	Repulsors
A	574	61 096	1 attractive fixed point	
A1	48	3 576	-1	—
A2	48	3 576	1	—
A3	37	3 026	-1	1
A4	37	3 026	1	-1
A5	113	12 720	$m^* \in]-1, 1[$	-1
A6	113	12 720	$m^* \in]-1, 1[$	1
A7	81	9 680	$m^* \in]-1, 1[$	$-1, 1$
A8	97	13 276	$m^* \in]-1, 1[$	$(-1, 1)$
B	31	444	2 attractive fixed points	
B1	1	36	$-1, 0^+$	$0^-, 1$
B2	1	36	$0^-, 1$	$-1, 0^+$
B3	1	15	$-1, 0^+$	0^-
B4	1	15	$1, 0^-$	0^+
B5	5	41	$-1, m_a^* \in]-1, 1[$	$m_r^* \in]-1, m_a^*[$
B6	5	41	$m_a^* \in]-1, 1[, 1$	$m_r^* \in]m_a^*, 1[$
B7	2	10	$-1, m_a^* \in]-1, 1[$	$m_r^* \in]-1, m_a^*[, 1$
B8	2	10	$m_a^* \in]-1, 1[, 1$	$-1, m_r^* \in]m_a^*, 1[$
B9	13	240	$-1, 1$	$m^* \in]-1, 1[$
C	67	2 018	1 attractive 2-cycle	
C1	8	32	$(m_1, m_2) \in]-1, 1[$	$-1, m^* \in]m_1, m_2[$
C2	8	32	$(m_1, m_2) \in]-1, 1[$	$m^* \in]m_1, m_2[, 1$
C3	43	1 854	$(-1, 1)$	$m^* \in]-1, 1[$
C4	8	100	$(m_1, m_2) \in]-1, 1[$	$m^* \in]m_1, m_2[, (-1, 1)$
D			1 attractive fixed point and 1 attractive 2-cycle	
	24	822	$(-1, 1), m_a^* \in]-1, 1[$	$(m_1, m_2) \in]-1, 1[$
E			2 attractive 2-cycles	
	2	12	$(-1, 1), (m_1, m_2) \in]-1, 1[$	$m^*, (m_1, m_2) \in]-1, 1[$
F	1	320	all m-values fixed points	
G	1	320	all m-values on 2-cycles $m \to -m$	
Total	700	65 536		

readily found as roots of a second-degree polynomial. This is the case for subclasses A3, A4, A7, B3, B4–B2, and F.

For example when $c_1 = -1$ and $c_5 = 1 m^* = -1$ and $m^* = 1$ are fixed points, and the other two solutions to the fixed point equation (8) are

$$m^* = \frac{4 + c_2 - c_4 \pm \sqrt{4(2 + c_2)(2 - c_4) + c_3^2}}{c_2 - c_3 + c_4}. \tag{11}$$

(If in (11) m^* does not belong to the interval $[-1, 1]$, it is not a fixed point of interest.) Maps T with this property make up the subclasses A3, A4, A7, B7–B2 and F, one quarter of the 700 maps under consideration.

The classes B1, B2, B3 and B4 have 0 as a doubly degenerate fixed point, attractive from one side, repulsive from the other, but *marginal,* with eigenvalue 1, to both sides. The approach to or exit from this fixed point is not geometric, but a case of critical slowing down, as that phenomenon is modelled by mean-field theory.

4. Relating mean-field results to the original problem

Equation (4) gives the exact result for $m(t+1)$ if the spins in the configuration $\{\sigma_i(t)\}_{i=1\dots N}$ are uncorrelated with magnetisation $m(t)$. The initial configuration $\{\sigma_i(0)\}_{i=1\dots N}$ is chosen that way; it has uncorrelated spins and a given magnetisation $m(0)$. So (4) gives the exact result for $m(1)$. The exact value of $m(2)$ is a function of $\{\sigma_i(1)\}_{i=1\dots N}$, and the groups of four spins that make up the input to the individual cellular automaton are correlated. This is where the mean-field *approximation* comes in, and is used in all ensuing time steps. We have observed that for many rules of f the first application of T to m gives the largest change of m towards its fixed-point or limit-cycle value. Since the choice of initial configuration makes mean-field theory exact in the first time step, our final classification may be more reliable than one might expect from the mere mean-field approximation.

Since $T(1)=f(1, 1, 1, 1)$ and $T(-1)=f(-1, -1, -1, -1)$, the mean-field description is exact whenever it predicts a fixed point or 2-cycle with magnetisation ± 1.

Before a general comparison can be made between mean-field and simulation results two things are needed: (i) simulation results, we await their publication [19]; (ii) an interpretation of mean-field and simulation results in terms of each other, this appears to be a trivial requisite, since the magnetisation is defined for both, and can be compared with no further ado. It is immaterial whether, for example, a fixed-point value determined in mean-field theory is realised in the simulation by a configuration that changes chaotically in time with constant magnetisation, or by a configuration dominated by spins of fixed value. There is a subtlety at play, however: the square lattice with nearest-neighbour interaction neglecting the central spin is decoupled into two sublattices such that the configuration on one sublattice at time $t+1$ depends only on the configuration on the other sublattice at time t. This means that mean-field results must be ascribed to each subsystem separately, and it is the *combined* behaviour of the two subsystems that can be compared with simulation results obtained by averaging over both subsystems. In general, both subsystems will obey the same mean-field dynamics, to the extent it is a reasonable approximation. The reason for this is that our initial configuration is chosen at random with a given magnetisation and no correlations between individual spins. It follows that so are the initial configurations of the two subsystems. Hence, in general they fall on the same attractor, and if it is a 2-cycle they have the same phase. But when the initial configuration is chosen with magnetisation equal or very close to that of a repulsive fixed point or cycle, fluctuations in a finite system being simulated may cause the two subsystems to end up on different attractors, or on the same 2-cycle, but with different phases. Consider, for example, the networks in class B2: they have 0^- and 1 as attractive fixed points, and 0^+ as repulsive fixed point. Starting a simulation with zero magnetisation, one subsystem may end up with magnetisation 0^-, the other with magnetisation 1, according to mean-field theory. But the total magnetisation is then 0.5, and seemingly way off the mean-field prediction for the class B2! The fact that 0 is a marginal fixed point is of no importance for the argument just presented. It holds with little change for all networks in classes B, C, D and E, and initial magnetisation on a repulsive point or cycle. Since mean-field theory is only an approximation one does not know *a priori* in a simulation exactly *where* these repulsive points and cycles are located, or for that matter which class a given network is in, or which classes there are. Hence, there is no way to avoid this complication except to simulate only one of the two subsystems. We recommend doing so, as it gives cleaner results with no loss of information or generality.

In principle, this presentation of results is only completed when we have given the relationship between all of the 700 mean-field classes and the classes in the table above, together with the magnetisations of the attractors and repulsors of all 700 maps. Even after eliminating one map in every pair related by (10) from the table, we are left with a table with more than 350 entries. In its entirety, such a table is meaningful only in machine-readable form, and can be obtained from the authors. For any particular automaton rule f the reader can easily find the class it belongs to with the help of this machine-readable table. The set of attractors and repulsors of any map T resulting from any rule f by (5) and (6) are listed in the table. Since we know that nothing more complicated than a 2-cycle is encountered, a crude plot of T suffices to decide the subclass it belongs to.

5. Conclusions

We have used dynamical mean-field theory to describe the time evolution of the magnetisation of each of the 65 536 different binary automata receiving four inputs, and classified them according to their limit behaviour. The mean-field description gave, by its mere application, a first classification of the 65 536 automata into 700 different classes.

For each of these classes the time evolution of the magnetisation $m(t)$ was given by a fourth-degree polynomial mapping $m(t)$ into $m(t+1)$. These 700 polynomials were classified according to the nature and location of fixed points and cycles of this map. That resulted in seven different classes. We have tabulated the fixed-point and cyclic values for the magnetisation, and made them available in machine-readable form.

We have described a complication encountered, when mean-field predictions are to be compared with simulation results, and recommended a solution requiring only a minor change in the way simulations are done for the networks discussed here.

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Note added in proof: We thank D Stauffer for providing us with the references: Stauffer D 1989 *Physica* **157A** 645; 1990 *Physica* **162A** 176; Gerling R W *Physica* **162A** 187; 196.

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